2019-Sep-10 (pre-lecture) Prof. Yun William Yu

Monday, September 9, 2019 11:24 AM

Outline:

1. On proofs & abstraction
   - What did we prove on Thursday?
   - Why is it important?
   - What are the different levels of solving a problem.

2. Explicit Solutions
   - First-order autonomous equations
   - Separable equations
   - ODEs with homogeneous coefficients
   - Exact differentials

What is a mathematical proof?

- A mathematical proof is a series of logically connected statements starting from what you know and ending with what you want to prove.

Thm of Apple Deliciousness

Claim: Apples are delicious.

Proof. It is self-evident that sweet foods are delicious. (starting axiom)

By the Theorem of Sugary Fruits, all fruits are sweet foods. (reference to another fact we agree on)

The computation below shows that apples are fruits. (computation)

Therefore, apples are delicious. $\square$ (conclusion)

- Each step must be justified by an agreed upon reason.
- The amount of justification needed and how much you can skip in your steps depends on your audience.

Once a claim is proved, you can use it in later proofs.

Claim: Oranges are delicious.

Proof. Oranges are apples. ← incorrect starting point

Therefore, oranges are fruit. ← true via the proof of Apple Deliciousness, given the incorrect prior statement

By the Theorem of Apple Deliciousness, oranges are delicious. ← used wrong theorem

$\square$

Recall: Consider a first-order autonomous equation.

$$\dot{x} = f(x) \quad , \quad f \in C(\mathbb{R}),$$

We want to find a function $\phi(t)$ s.t.

$$\dot{x}(\phi(t)) = f(\phi(t)).$$

} i.e. find a general solution to ODEs of this type.

Because it is time-invariant, we need only consider solutions starting at $x_0 = x(0)$.

We proved that if $f(x_0) \neq 0$, we can define a function $F(x) = \int_{x_0}^{x} \frac{dy}{f(y)}$ in some interval of validity (x_1, x_2),

where $f(x) \neq 0 \quad \forall x \in (x_1, x_2)$,

$\Rightarrow F(x)$ is strictly monotonic.

Why did we define $F(x)$ and prove strict monotonicity?

Suppose $\phi(t)$ is a solution to the ODE $\dot{x} = f(x)$.

Then $F(\phi(t)) = \int_{x_0}^{\phi(t)} \frac{dy}{f(y)} = t$ (from Thursday's proof.)

$$\left(\int_{x_0}^{\phi(t)} \frac{dy}{f(y)} = \int_0^t \frac{\dot{\phi}(s)}{f(\phi(s))} ds = \int_0^t ds = t \right)$$

If $F(\phi(t)) = t$, then $\phi(t) = F^{-1}(t)$ so long as F^{-1} exists.

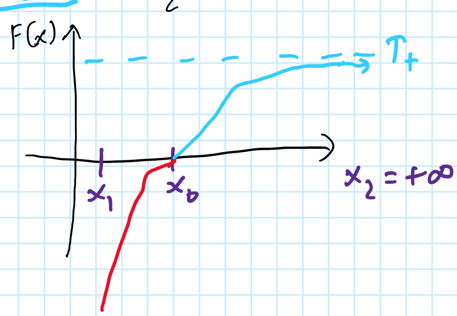
F^{-1} exists so long as $F(x)$ is strictly monotonic.

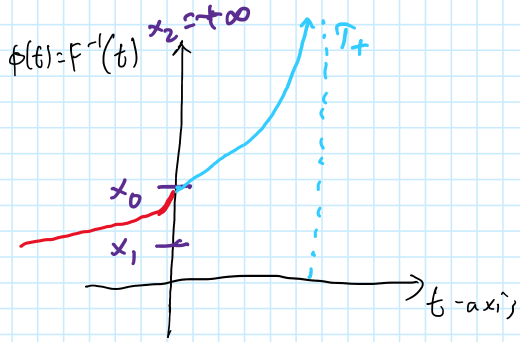
This proves that we can find a solution by integrating if $f(x_0) \neq 0$.

But these solutions are only valid on (x_1, x_2). What happens as we get closer to those limits? Let's consider x_2 for $F(x)$ strictly positive monotonic.

Recall $T_+ = \lim_{x \uparrow x_2} F(x) \quad \left(= \lim_{x \to x_2^-} F(x) \right).$

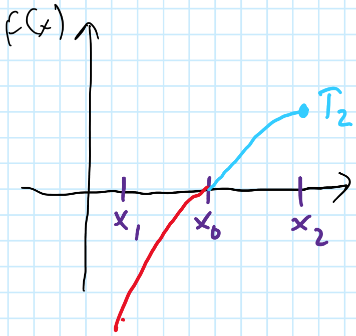
Case 1: $x_2 = +\infty$



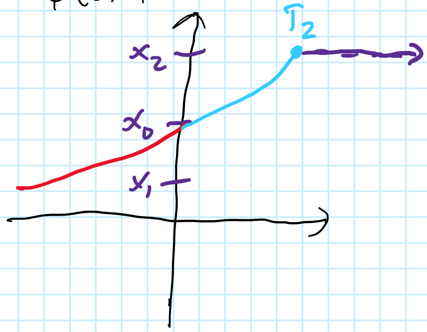


$\phi(t)$ asymptotically approaches $+\infty$, as $t \uparrow T_+$, so the function cannot be extended.

Case 2: $x_2 < +\infty$



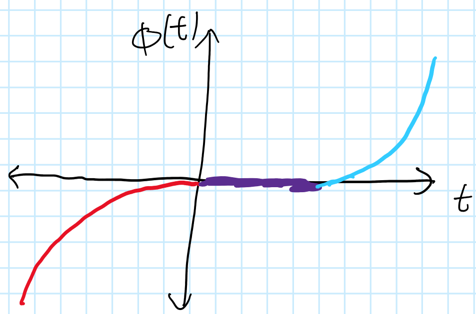
$\phi(t) = F^{-1}(t)$



$\phi(t)$ approaches a finite T_2, and $f(x_2) = 0$, so can extend the constant solution.

Odd behavior can appear when $f(x) = 0$

Ex. $f(x) = \sqrt{|x|}$, $x_0 = 0$. $\phi(t) = \begin{cases} -\frac{(t-t_0)^2}{4}, & t \le t_0 \\ 0, & t_0 \le t \le t_1, \\ \frac{(t-t_1)^2}{4}, & t_1 \le t \end{cases}$ (see Example 1.33, Teschl)



Conclusion: • We can usually integrate to get solutions for 1st order autonomous ODEs.

• But solutions might only exist locally in t.

• And, solutions might not be unique.

Levels of Abstraction:

More abstract/general

$F(t, x, x', x'', \dots, x^{(k)}) = 0$ (general functional relationship)

$x^{(k)} = f(t, x, x', \dots, x^{(k-1)})$ (explicit form kth order)

first order $(x' = f(t,x))$, second order $(x'' = f(t, x, x'))$, ...

first order autonomous $(\dot{x} = f(x))$, separable $(\dot{x} = g(t) f(x))$, ...

$\dot{x} = -kx$, $\dot{x} = ax + b$, $\dot{x} = c_1 \sin(x) + c_2 \cos(x)$, ...

$\dot{x} = -5x$, $\dot{x} = 5x + 2$, ...

radiocarbon dating, compound interest, falling under gravitational field.

More specific/applied

We just proved we can solve first order autonomous equations by integrating.

We just proved we can solve first order autonomous equations by integrating.

Explicit Solutions: We just gave a general solution for one class of ODEs. This and other general solutions allow us to solve specific problems.

Example: $\dot{x} = x^2, \quad x(0) = x_0$

$$\frac{dx}{dt} = x^2$$

If $x \neq 0$, $\quad \frac{dx}{x^2} = dt$

$$\int \frac{dx}{x^2} = \int dt$$

$$-\frac{1}{x} = t + C$$

$$\frac{1}{x} = -t + C$$

$$x = \frac{1}{-t + C}$$

$$x(0) = \frac{1}{C} = x_0$$

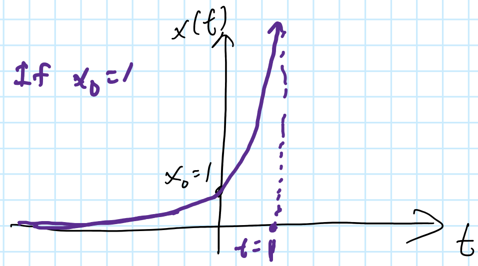
$$\Rightarrow C = \frac{1}{x_0}$$

$$x = \frac{1}{-t + \frac{1}{x_0}} = \frac{x_0}{1 - x_0 t}$$

Including explicit transformations for constants

$$\left(\begin{array}{l} -\frac{1}{x} = t + C_1 \\ \frac{1}{x} = -t + C_2, \quad C_2 = -C_1 \\ x = \frac{1}{-t + C_2} \end{array} \right)$$

If $x_0 = 1$



Note: Whenever we do integration, we often end up with not just one solution, but an entire family of them because of constants.

Define: A *k-parameter* family of solutions for an ODE to be a solution that has k constant terms $C_1, \dots, C_k$.

Ex. $f'''(t) = 5$

$$\int f'''(t)\,dt = \int 5\,dt$$

$$f''(t) = 5t + C_1$$

$$f'(t) = \frac{5}{2}t^2 + C_1 t + C_2$$

$$f(t) = \frac{5}{6}t^3 + \frac{1}{2}C_1 t^2 + C_2 t + C_3 \quad \leftarrow \text{3-parameter family}$$

Ex. (from above) $\quad \dot{x}(t) = x^2(t)$

Ex. (from above) $\dot{x}(t) = x^2(t)$

$$x(t) = \frac{1}{-t+c} \leftarrow \text{1-parameter family}$$

Ex. $y'' - y = 0$

$\Rightarrow y = C_1 e^{x} + C_2 e^{-x}$

↑ 2-parameter family

We haven't learned to solve this yet, but can check

$y' = C_1 e^{x} - C_2 e^{-x}$

$y'' = C_1 e^{x} + C_2 e^{-x} = y.$

For many (but not all) ODEs, we can say that a kth-order ODE has a k-parameter family of solutions.

Def. A particular solution as a solution that has no arbitrary constants.

Def. A general solution is a k-parameter family of solutions that contains every particular solution of an ODE.

Using this terminology, we say that we can find a 1-parameter family of solutions to a 1st-order autonomous ODE by integrating. If for $\dot{x} = f(x)$, $f(x) \neq 0$ anywhere in the domain, that 1-parameter family of solutions is a general solution.

Other types of ODEs.

Separable coefficients

Define: A separable first-order ODE is an ODE that can be rewritten

$$\dot{x}(t, x) = g(t) f(x). \quad \left(\text{or equivalently } g(t)dt + f(x)dx = 0\right)$$

We can solve separable ODEs using the same ruse.

$$\frac{dx}{dt} = g(t) f(x)$$

$$\int \frac{dx}{f(x)} = \int g(t)\,dt, \quad \text{so long as } f(x) \neq 0.$$

Ex. $\dot{x}(t, x) = \sin(t) \cdot (x+1)$

$$\frac{dx}{dt} = \sin(t) \cdot (x+1)$$

$$\frac{dx}{x+1} = \sin(t)\,dt$$

$$\int \frac{dx}{x+1} = \int \sin(t)\,dt$$

If $x > -1$, $\ln(x+1) = -\cos(t) + C$
$$x+1 = e^{-\cos(t)+C} = Ce^{-\cos t}$$
$$x = -1 + Ce^{-\cos(t)}$$

If $x < -1$, $\ln(-x-1) = -\cos(t) + C$
$$-x-1 = Ce^{-\cos(t)}$$
$$x = -1 - Ce^{-\cos(t)}$$

$\Rightarrow$ If $x \neq -1$, $x = -1 + Ce^{-\cos(t)}$

If $x = -1$, we guess that $x = -1$ is also a constant solution, because then $\dot{x}(t,x) = 0$.

So now we have a 1-parameter family of solutions $x = -1 + Ce^{-\cos(t)}$.

Ex. 6.7 Tenenbaum

Find a particular solution of $xy^2\,dx + (1-x)\,dy = 0$ for which $y(2) = 1$.

Let's rewrite this in a separated form

$$xy^2\,dx + (1-x)\,dy = 0$$

If $y \neq 0$ and $x \neq 1$,
$$\frac{x}{1-x}\,dx + \frac{1}{y^2}\,dy = 0$$

$$\int \frac{x}{1-x}\,dx + \int \frac{1}{y^2}\,dy = C$$

$$\int \left(\frac{1}{1-x} - 1\right)dx + \int \frac{1}{y^2}\,dy = C$$

$$\int \frac{1}{1-x}\,dx - \int dx + \int \frac{1}{y^2}\,dy = C$$

$$-\ln|1-x| - x - \frac{1}{y} = C$$

$$\ln|1-x| + x + \frac{1}{y} = C$$ $\leftarrow$ 1 parameter family of solutions (implicit)

Recall $y(2) = 1$ (i.e. $x=2, y=1$ is a point on the curve of the particular solution)

$$\ln|1-2| + 2 + 1 = C$$
$$C = 3$$

$\Rightarrow$ Particular solution: $\ln|1-x| + x + \frac{1}{y} = 3$, $x \neq 1$, $y \neq 0$

Homogeneous coefficients - do not confuse with homogeneous linear ODEs which don't have a constant term from

Homogeneous coefficients – do not confuse with homogeneous linear ODEs which don't have a constant term from Lecture 1.

Def. Let $z = f(x,y)$ be a function of x and y.
$f(x,y)$ is said to be **homogeneous of order n** if it can be written as $f(x,y) = x^n g(u)$, where $u = \frac{y}{x}$

or $f(x,y) = y^n g(u)$, where $u = \frac{x}{y}$.

Ex. $f(x,y) = x^2 + y^2 \log \frac{y}{x}$, $x > 0$, $y > 0$.

$$f(x,y) = x^2\left(1 + \frac{y^2}{x^2}\log\frac{y}{x}\right).$$

Substituting in $u = \frac{y}{x}$, $f(x,y) = x^2\left(1 + u^2 \log u\right)$

Thus, $x^2 + y^2 \log \frac{y}{x}$ is homogeneous of order 2.

Alternately,

$$f(x,y) = y^2\left(\frac{x^2}{y^2} + \log\frac{y}{x}\right)$$

Let $u = \frac{x}{y}$. $f(x,y) = y^2\left(u^2 + \log\frac{1}{u}\right) = y^2\left(u^2 - \log u\right)$.

So it is still homogeneous of order 2.

Alternately, an equivalent definition is that a function $f(x,y)$ is **homogeneous of order n** if

$$f(tx, ty) = t^n f(x,y).$$

Ex. $f(x,y) = x^2 + y^2 \log \frac{y}{x}$.

$$f(tx, ty) = t^2 x^2 + t^2 y^2 \log\frac{tx}{ty} = t^2\left(x^2 + y^2 \log\frac{y}{x}\right) = t^2 f(x,y)$$

Mentimeter: Are the following functions homogeneous? If so, what order?

1. $e^{y/x} + \tan\left(\frac{y}{x}\right)$
2. $x^2 + \sin x \cos y$
3. $\sqrt{x+y}$
4.

3. $\sqrt{x} + y$

4. $\sqrt{x^2 + 3xy + 2y^2}$

5. $x^4 - 3x^3y + 5y^2x^2 - 2y^4$

Definition 7.2 (Tenenbaum) The differential equation

$$P(x,y)\,dx + Q(x,y)\,dy = 0,$$

where $P(x,y)$ and $Q(x,y)$ are each homogeneous functions of order n is called a **first order differential equation with homogeneous coefficients.**

Theorem 7.32 (Tenenbaum) Given an ODE $P(x,y)dx + Q(x,y)dy = 0$ with homogeneous coefficients, the substitution

$$y = ux, \quad dy = u\,dx + x\,du$$

leads to a separable ODE in u and x.

Ex. $2ye^{x/y}\,dx + (y - 2xe^{x/y})\,dy = 0$

Note $2ye^{x/y}$ is homogeneous with order 1

$y - 2xe^{x/y}$ is homogeneous with order 1.

Let $x = uy, \quad dx = u\,dy + y\,du$

$$2ye^u(u\,dy + y\,du) + (y - 2uye^u)\,dy = 0$$

$$\underline{2uye^u\,dy} + 2y^2e^u\,du + y\,dy - \underline{2uye^u\,dy} = 0$$

$$2y^2e^u\,du + y\,dy = 0$$

$$2e^u\,du + \frac{1}{y}\,dy = 0$$

$$2e^u + \ln|y| = C$$

$$2e^{x/y} + \ln|y| = C$$

Exact differentials

From multivariable calculus (the coreq B41), we can define the total differential of a function $z = f(x,y)$ by

$$dz = \frac{\partial f(x,y)}{\partial x}\,dx + \frac{\partial f(x,y)}{\partial y}\,dy.$$

$$dz = \frac{\partial \ldots}{\partial x} dx + \frac{\partial \ldots}{\partial y} dy.$$

Ex. $z = x^2 + y^2$

$dz = 2x\,dx + 2y\,dy$

Def. A differential expression

$$P(x,y)dx + Q(x,y)dy$$

is called an **exact differential** if it is the total differential of some function $f(x,y)$.

i.e. if $P(x,y) = \frac{\partial}{\partial x} f(x,y)$ and $Q(x,y) = \frac{\partial}{\partial y} f(x,y)$.

When an exact differential appears in an ODE, and if we know the function it is the total differential of, then we can easily integrate the **exact differential equation** to get a 1-parameter family of solutions

$$f(x,y) = c.$$

Ex. $(6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy = 0$

The LHS is an exact differential, the total differential of $f(x,y) = 3x^2y + 5xy + y^3$.

Thus $3x^2y + 5xy + y^3 = C$ is a solution to the ODE.

How do we recognize exact differentials?

Theorem 9.3 (Tenenbaum): $P(x,y)dx + Q(x,y)dy = 0$ is exact if and only if

$$\frac{\partial}{\partial y} P(x,y) = \frac{\partial}{\partial x} Q(x,y),$$

where $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ all exist and are continuous in a simply connected region $R \subseteq \mathbb{R}^2$.

Proof. See Lesson 9B, Tenenbaum.

Ex. $2x\,dx + 3y\,dy = 0$

$$\frac{\partial}{\partial y}(2x) = 0 \qquad \frac{\partial}{\partial x}(3y) = 0. \qquad \text{Exact}$$

$$(2x + \sin y)\,dx + (x\cos y + y^2 - 5y)\,dy = 0 \qquad \text{Exact}$$

$$\frac{\partial}{\partial y}(2x + \sin y) = \cos y \qquad \frac{\partial}{\partial x}(x\cos y + y^2 - 5y) = \cos y$$

Mentimeter

$$\left(\frac{2xy+1}{y}\right)dx + \left(\frac{y-x}{y^2}\right)dy = 0$$